Abstract: This paper constructs the probability space underlying the random variable of any time dependent econometric specification. The construction links concrete economic activity, both perceived and recorded, to econometric formulations. Furthermore, it is argued that the probability events belonging to this space are forms of understanding economic activity held by each agent. The model establishes two aspects of any econometric formulation. Mainly, that learning must be unique between any two ticks of the clock and that not all forms of understandings can indeed become events in the random variable’s probability space.
JEL classification: B41
Keywords: knowledge, probability spaces, topology, learning, heterogeneity.
**Introduction**

The aim of this paper is to describe and develop a mathematical model of the formation and constitution of the probability space associated to the individual disturbance factor $\lambda_t$ of any time dependent statistical system. It is argued that this probability space is indeed constituted by events of knowledge. The construction of this space, termed understandings space, rests on the links between past economic activity and the agents’ ability to understand economic activity. For this, a detailed exploration into the nature and constitution of learning, as defined in here, is done. This requires some basic definitions on the nature of knowledge to be employed in here. Moreover, the role of recorded information is highlighted as a conditioning factor of learning.

Knowledge and intuitions are characterised as abstract mathematical entities, just as other approaches have\(^1\). Our approach, however, is based on elementary set theory and topology. Individual understandings are linked to concrete records of economic activity through formal operators, i.e. dialectics, learning and probabilities. In order to technically construct the agent’s probability measure, certain assumptions, to be spelled later on, are made in order to invoke the Riesz Representation Theorem (RRT). This approach is only one of possibly many constructions. We purposely chose it in order to highlight the core aspects of the dynamics of understandings. That is, to bring history to the front of knowledge, and intuition generation within a concrete model. The entire description will be done for one random individual agent.

The paper begins with a set of basic definitions, i.e. knowledge, intuitions and understandings. Then, in the next section, it describes the basic constituents of the dynamic process of understandings generation and their mathematical formulation. Furthermore, it establishes the basic abstract spaces and the corresponding operators that link them. General results are presented to provide additional characterisations of these spaces. The next section is entirely devoted to a detailed account of the nature of learning, as defined here, and the construction of the probability measure associated to the disturbance term of any time series statistical system. Finally, conclusions are drawn in the last section.

### I Knowledge, intuitions and understandings

In any capitalist market with at least two interacting economic agents, individual beliefs are always based on forms of understandings at least partially dependent on the others’ understandings, beliefs and expected behaviour. In the economics literature of knowledge this can be traced back to, at least, Von Hayek (1937). Game theory, particularly evolutionary game theory, relies heavily on this feature of individual behaviour, e.g. Weibull (1995), Mailath (1998). Furthermore, those expressions of understandings relevant to the agents are acquired through particular idiosyncratic processes, i.e. learning, in each case unique to the agent, e.g. Loasby (1999), Slembeck (1999). Learning, in this sense, allows individual economic agents to transform economic history into beliefs and expectations of economic activity. Whence, this begs the question of what knowledge is. Although many different definitions and approaches can be invoked, the following will be used for the purposes of the present paper:

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1 For example, the mathematical psychology and game theory literatures.
Knowledge. We assume the perspective of the materialistic theory of knowledge. Knowledge is the understood (or apprehended) portion of the synthesis of a materially based dialectical contradiction.

Intuitions. These are forms of knowledge as well. They are non-dialectical and originate from reality itself. Formally, it can be defined as a direct relation between the mind and something abstract, therefore, not accessible through the senses (Oxford Companion of Philosophy).

Understandings. These can be either knowledge or intuitions.

Our definitions allow us to highlight the totally subjective nature of the economic agents’ understandings. In this sense, the approach to be followed in the paper is based on the idea of events of understandings rather than problem-solving abilities. It is more or less in the spirit of Hintikka (1962) but different in that the main emphasis is placed on the inner structure of the understanding spaces rather than in the construction of systems of understandings through logical operators. More importantly, understandings, as defined here, are always and anywhere an abstraction. It follows naturally to represent them mathematically as abstract entities, i.e. abstract points, in some space of events of understandings. This is done in the next sections.

Any examination of an agent’s understandings necessarily requires a consideration on time. As time develops, whatever its meaning and notion of, economic history unravels and agents (at least attempt to) understand it. Time, history and understandings are indeed unavoidably inseparable. If time were already eternal, history would cease to exist and there would be nothing further to discover and understand, Metcalfe (1998). Moreover, if there were no time, not only would economic agents not grasp the entirety of their surrounding reality (because human learning is not enough to discover all of the forces that drive economic activity) but also they would have no possibility to expand their horizon of understanding. Only Shacklian time would continue to exist as a reflection of individual “silent contemplation”. In such a situation, understandings would reduce themselves to a set of rules and recipes upon which routines can follow into perpetuity. A time-independent attractor will have been reached in which economic activity will have been reduced to timeless, unchanging, economic patterns. Without change in time there cannot be change in the facts of history beyond the equilibrated activity of historical attractors; understandings would also remain in the same resilient state of affairs. Whence, change (beyond that of stable equilibrium) in any one of the three variables, i.e. time, history and understandings, necessarily implies change in the other two. In this sense, only past dependent processes can induce new forms of understandings, Metcalfe (1998).

Human limitations, e.g. of storage, learning, searching and processing capacities, etc., allow the possibility of further knowledge to be pursued at all time, Loasby (1999). In particular, if self-consistent propositional systems are constructed to generate knowledge, it can be proven that such systems are always incomplete and hence the knowledge derived thereof is incomplete as well. The consequences of this theorem, rightly called Gödel’s theorem, are immensely profound and are certainly

2 The full description of Hintikka’s approach can be found in Rubinstein (1998)’s Modelling Bounded Rationality.
beyond the scope of this work. However, one corollary (of this theorem) does have
direct bearings on this work. If the systems of propositions constructed through
binary logic are always incomplete then knowledge is always potentially expandable
although in a limited boundedly rational fashion, e.g. Morgenstern (1935), Koppl and
Rosser (2000). That is, there is always the possibility that additional yet-
 undiscovered knowledge and intuitions can influence recorded economic activity as a
matter of empirical evidence. Therefore, the very own non-quantifiable nature of
knowledge and intuitions implies that, at any point in time, it is the disturbance term,
i.e. $\lambda_i$, of any time dependent econometric formulation such as

$$
 h_t = f(h_{t-1}, z_{t-1}) + \lambda_t
$$

the only measurable channel of influences and dependencies based on understandings.
If the modeller did know it all then he/she could specify each and every influence that
affected him/her (through $h^{t-1}$ or $z^{t-1}$). Whence, there would be no need for a
disturbance term in the specification of his data generating mechanism. In this sense,
the consequences of the modeller’s cognitive limitations (and the agent’s as well)
justify the existence of the random variable in the specification above.

There already exists a huge literature in rational choice theory and
mathematical psychology on the construction of understandings spaces. The
traditional mathematical psychology literature conceptualises understandings spaces
in relation to problem solving schema. This general conception of understandings is
not really concerned with knowledge’s (nor intuitions’) origin. This view faces
further additional difficulties in that it does not take a deep concern as to the effects of
individual search for knowledge. In all fairness, though, this approach, which is
based on behaviourist psychology, is not aimed at studying knowledge in itself but
rather how it is handled and dealt with. Hence, the difference in the type of agents,
e.g. atomistic unintended neoclassical agents, Schumpeterian individual entrepreneurs
or Penrosian conglomerates of peoples gathered in firms, is not in question.
Coordination is not generally an issue either. However, in as far as understandings
are concerned this is not a trivial matter for clearly, problem solving abilities differ
from individuals to uncoordinated groups of people to coordinated groups of people.
In any case, this literature defines the set of all understandings states as the agent’s
knowledge structure for the particular set of questions. If this understandings
structure is closed under arbitrary unions then it is called the agent’s knowledge space
for the set of questions. Further structures of order can be considered if it is also
closed under intersections. A thorough exposition of this approach can be found in
Albert, Schrepp and Held (1994) and Doignon and Falmagne (1998). In particular,
Suck (1999) provides an interesting metric for discrete knowledge spaces which, in
principle, allows the definition of distance of understandings among agents. This
distance is constructed on a “lattice of understandings”. That is, given certain tasks

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3 Some very interesting research has been done in this respect in related areas. For example,
Wolpert (1996) presents a theorem on the impossibility of building a machine capable of calculating
future specifications of a physical system. Hut, Ruelle and Traub (1998) study the limits to knowledge
in physics and biology. Minkler (1993) argues on the limits that dispersed knowledge creates for
modelling and managerial purposes; he proposes a core capability approach in stead. Langlois (2001)
reviews the main concepts of rationality theory and suggests that institutions shape the emergence of
rules and routines undertaken by agents. Although not stated in his paper, this is a clear statement on
the limitations of human searching mechanisms through which institutions constrain as much as they
stimulate further knowledge. Loasby (2000) argues the same point on the role of institutions in the
question of preference formation in demand in an evolving market.

4 This is our understanding space.
there will exist certain answers, however good or bad. Hence, these answers can be mapped onto an ordered set of points in a vector space, i.e. a lattice. Through this measure, the differences, i.e. the distances, in the possibilities of knowledge between two spaces based on recorded know-how can be estimated. Therefore, the consequences of the differences in problem solving abilities may be estimated.

II The Dynamics of Understandings Formation

When the agent is considered at time \( t \) he/she, in reality, faces, as a constituting member of the industry, a given history of relevant facts in the industry, a sort of factography of the industry. This factography is objective in nature as is encapsulated information, e.g. letters, numbers, equations, combinations thereof, etc. Therefore, objective phenomena, i.e. data, are in principle common to all agents although not necessarily true\(^5\). In fact, at \( t \), the agent has more than just a given accumulated history of data; he also inherits accumulated, concentrated and clustered expressions of past reconstructable understandings related to information up to \( t \). With the sole purpose of enhancing his/her position in the market he/she will necessarily strive for further understanding\(^6\). Possible future economic activity pulls (past) history to the front, very much, as Von Hayek (1937) and Loasby (1999) imply it and Marx (1867) and Morgenstern (1935) clearly state it.

Consider an interval of time during which economic activity develops. History’s own changing constitution will induce changes upon the available recorded facts of history. Simultaneously, changes in economic activity, determined by the evolving market forces, will induce changes upon the systems of dialectical contradictions that so reflect economic activity as well. Hence, what is at the agents’ disposal, as a reservoir of understandings, will change as a consequence of the existence of history. In other words, the domain of the entities to be learnt, i.e. syntheses, will change due to an developing environment. Distinct instants of time have different systems of dialectical contradictions\(^7\) associated to them and hence different dialectical realities emerge; ultimately, due to history never repeating itself. Conceptually, in this context, dialectical materialism is a dynamic operator defined over material economic reality. This will be defined later on.

As history develops itself, new potential forms of understandings emerge, which, with the aid of information, amongst other things, are apprehended through learning. Once learning has taken place, the agents hold forms of understandings, whether knowledge or intuitions, related to their surrounding economic activity. These heterogeneous forms of understanding, ultimately, supply and demand, generate differences across firms and constitute uneven core capabilities in the market as in Nelson (1991).

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\(^5\) This is the issue of asymmetric information. Although not explicitly dealt with here it certainly constitutes an important part of the dynamics described here. Essentially, available information partly determines individual learning. For an introduction to problems of asymmetric information in economics see Hillier (1997) and of course Stigler (1961).

\(^6\) The aggregated outcomes of the individual intent in the search for understanding will be the unintended resulting economic activity, as observed by Adam Smith (1776). That is, the invisible coordinating hand.

\(^7\) A system of dialectical contradictions at any moment in time should be understood as being composed of two categories of contradictions: a principal contradiction and peripheral contradictions. The principal contradiction is the contradiction that allows for the existence of the phenomenon in question. Peripheral, or secondary, contradictions define the particular expression of the phenomenon.
To summarise: at any moment in time, learning takes place formally from systems of dialectical contradictions and delivers knowledge. Informal learning operates over reality itself and delivers intuitions. Data banks partially influence both forms of learning in a positively correlated manner. At any moment in time, given certain requirements on learning, a probability distribution for the disturbance term of (any) the time dependent econometric formulation can be constructed (this will be proven later on). The next recorded facts of history will carry the influences of present understandings, channelled first through learning and then the probability distribution. These data will constitute (a very small) part of the recorded economic activity in the future. Future economic activity will have fed on itself and the spiral of history will have induced a self-feeding loop of cognition in the agents.

III The Role of Learning
Learning plays a fundamental role in any evolving and adapting process of interacting heterogeneous agents, Von Hayek (1937), Metcalfe (1998), Loasby (1999). It is not only a process of observation, experience, internalisation and ultimate adaptation. It is also a process of discovery (in the Austrian sense, see Kirzner (1994, 1997)) and reconnaissance, or as Shackle (1961) calls it, silent observation. This is so since, through time the agents, amongst other things, in fact recognise, classify and categorise their surroundings. They gather and associate relevant phenomena around them. As Loasby (1999, 2000, 2001) has repeatedly emphasised, agents relate perceived economic phenomena through different mental processes. Furthermore, through this process of reconnaissance they can indistinctly and inductively further discover the forces that drive economic activity. In this respect, discovering new dimensions of reality, just like discovering new emergent markets, inevitably induces some sort of adaptation of the means by which learning takes place.

Two issues arise in this respect. The first one is that learning is an evolving adapting process itself. It responds to an evolving track record on the competence with which the agent learns and incorporates new experiences of cognition. In other words, changes in learning are a reflection on the past success of the agent’s mechanisms and procedures used in the past to enhance his/her understandings about the industry. In other words, it is a reflection on how far he/she has fulfilled his/her cognitive potential. In this sense agents learn to learn over time. This is what the management literature calls “double learning” or “double loop”. The reasons for the changes in the manner in which agents learn, and this is the second issue, imply that learning need not be unique at all. In other words, an agent could possibly learn through more than one methodology at the same time if so desired. Whence, in principle, learning is not a sole unique time-dependent process and each of its multifaceted forms evolves and adapts through time. This view is concomitant with the tradition, which conceptualises learning as a dynamic process, many times a repeated process, found in dynamic and evolutionary game theory; see Slembeck (1999) and Weibull (1995). In other words, the passage from history to understanding spaces is, in principle, as n-fold as the n procedures by which the agent formally learns.

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8 For example, the agent can repeat his/her previous procedure or he/she can innovate and incur in new knowledge acquiring techniques or imitate what is perceived as successful, etc. All three situations differ from the previous attempt to learn and represent at least three different methods to learn from the past. Other methods of learning have indeed been identified in the literature, for details see Slembeck (1999).
To further blur the distinctive features of individual learning, intuitions only reinforce the idiosyncraticity of understandings. Indeed, in our context, intuitions have only one recognisable characteristic: they always represent forms of understanding. An experience of intuition is always an experience of cognition. They help, aim and ultimately influence belief formation and decisions concerning the individual and collective cognitive characteristics of economic phenomena. More importantly, the very nature of intuitions does not permit for a specification on the mechanism through which they emerge.

Dialectics defines syntheses from which understandings sprout by identifying opposing forces. If all of the forces were identified then the continuum of forces would be known. Hence, reality would, in this situation, be a continuum of knowledge. Nevertheless, as Penrose (1959), Lachmann (1976) and Polanyi (1946) hint and Von Mises (1957) and Popper (1950, 1956) clearly state, economic agents are limited in their search to know the world. They are constrained because of limited abilities to know the world (Penrose (1959)) and because the world is largely unknowable, Lachmann (1976), Polanyi (1946), Von Mises (1957)\(^9\), Popper (1950, 1956). In this sense, as far as economic agents are concerned, knowledge is never complete since we cannot know it all.

In reality, if the mechanisms by which intuitions are apprehended were known, then their emergence could be framed in terms of formal learning. This is the reason for our definition of intuitions at the beginning\(^10\). In order to retain consistency with our entire exposition and also the belief on the complete inadequacy of idealistic understandings, we are restricting ourselves to intuitions emanated only from material economic activity. In the next section this will be defined formally.

Since intuitions, because of their very nature, cannot be constructed as self-consistent systems of propositions, Gödel’s theorem does not induce any limitation to their scope. Indeed, in principle, there is no limit to intuitions although they are bounded by human senses. Furthermore, limitation on intuitions would require restrictions (of logic or epistemology) that do not exist in the agent’s mental intuitive processes. They are not a subject of matter simply because if the questions of logic or epistemology ever became an issue then intuitions would not be intuitions any more. For logic or epistemology to become issues, intuitions must be conceptualised as dialectical phenomena, that is, (in the present scheme) as knowledge. The moment an intuition is framed into dialectics then all previously mentioned issues on the limits of dialectics repeat themselves. Whence, if intuitions are boundless, because of their non-dialectical nature, then they can, in principle, converge to the limit of possible understandings of economic activity, i.e. the continuum of cognition. In this sense it

\(^9\) Von Mises (1957), pp. 8-9, develops a further argument on the limitations to human knowledge. In it, he expands on the idea that what we can know is limited by the universe in which our senses operate. That is, we can only know the universe of which our senses tell us that we are a part of. It is intrinsically Kant’s argument on the limited use of pure-reason. In his own words: “Human knowledge is conditioned by the power of the human mind and the extent of the sphere in which objects evoke human sensations... There may also exist outside of the orbit we call the universe other systems of things about which we cannot learn anything because, for the time being, no traces of their existence penetrate into our sphere in a way that can modify our sensations.” In other words, there may be universes that we have not yet discovered and hence do not know of.

\(^10\) That is, intuitions are the outcome of yet-to-be discovered protocols of learning; protocols that we have called informal learning in this paper.
is that intuitions may be infinite (not in that there may be an infinite amount of them). In as far as economic agents, only intuitions can hence guarantee the possibility of understanding the infinitely dense dimensions of economic reality. In this sense as well, if an economic agent ever desired to further know economic activity, intuitions present themselves as a guiding mechanism, a compass of cognition of sorts, in the search for further knowledge. For our descriptive purposes, it represents one of the components of the two-fold process of individual learning.

IV The Model
In order to formalise mathematically some of the notions of the dynamics of understandings presented above, it will be necessary to make certain assumptions; in particular, with respect to history and systems of dialectical contradictions. However, none of the assumptions concerning their mathematical representation undermine the description. Nor is the essence of the argument compromised. They are all, basically, requirements of formalisation. Each one of these will be carefully explained whenever necessary.

History and “Factography”
Assume that time is measured discretely (and hence countable) and that measurements of economic activity take place in this frame. Assume that history is a continuous space irrespective of the measurements of time associated to it. Call this space of history $H_t$ where the subscript $t$ merely signifies that it is history considered up to time $t$.

Associated to the passage of history, records of it arise. Indeed, assume that all facts of history can be stored and arrayed in some manner so that these recorded facts, i.e. factography, can be grouped as

$$F_t = \{f_j : f_j \text{ is a recorded fact of history at time } j \leq t\}$$

Furthermore, assume that $F_t$ is countable and that the cardinality of $F_t$ is less than or equal than the first infinitely countable cardinal, i.e. $\aleph_0$, for any $t$, i.e. $\mathcal{C}(F_t) \leq \aleph_0$.

This set varies with time, as new facts of history are included with the passage of time. It may or may not be connected (in the topological sense) depending on the manner in which data is stored and arrayed. Operations amongst the elements of $F_t$, i.e. essentially unions and intersections, are always well defined provided that there exists a well-defined form of storage of information, e.g. bits, symbols, numbers, functions, etc. Once coherence in the storage of records has been achieved, handling the data is a matter of practical concern only. For the argument’s sake, if need be, simply assume that there exists a well-defined self-consistent storage mechanism that allows data to be handled.

Systems Of Dialectical Contradictions
Dialectical contradictions are, in essence, abstractions, that reflect opposing forces present in concrete economic activity, i.e. unities of opposing forces. When these unities of opposites are concatenated they form a system. Thus, let

$$K_t = \{k : k \text{ is a dialectical contradiction and } k \leftrightarrow (a,b) \text{ where } a \text{ and } b \text{ are the two opposing forces within concrete material economic activity that define } k\}$$

[2.a]

In other words, each element of $K_t$ is an abstract point in a set defined by two opposing forces from concrete economic activity. The definition above in no way
should be understood to mean that these are vectors or 2-ple.

Note first that the set $K_t$ is also time dependent for as history induces change on economic activity so does economic activity on the system that reflects it. Second, note that this set is connected for all $t$. The reason is simple: the syntheses of each of the dialectical contradictions concatenate, i.e. bind, the system as one single, inseparable abstract entity that reflects all of the relevant economic reality at once. If the system were not concatenated as one inseparable entity then there would exist two non-intersecting sets of syntheses of dialectical contradictions $K^1_t$, $K^2_t$, i.e. $K^1_t \cap K^2_t = \phi$. That is, dialectical materialism would be reflecting two parallel material realities. This is, by any consideration of individual cognition, wholly absurd for it can only be a pathological cognitive scenario (two parallel realities are, by definition, a schizophrenic world). Reality, is one and inseparable, at any point in time.

**Possible And Probable Understandings**

If an agent is to always possess the possibility of coming up with a revolutionary understanding, capable of perhaps changing the whole industry, then such an event of understanding must be present in the understanding space, even if it possesses only a very small probability. Otherwise, the possibility of novelty could not, in general, be guaranteed. In practice, psychological or cultural attributes are probably the most deterministic characteristics in the agent’s ability to discern the surrounding economic activity. However, for modelling purposes, it is necessary to maintain the very same possibility as a probable event. In that manner it could always happen, as any external modeller should expect. Ultimately, culture and individual psychological development will condition individual understandings but only probabilities will guarantee novelty.

For expositional purposes the underlying understanding space will first be defined and then, in the next section, constructed. In this particular context, it means that $\lambda^1_i$ is a measurable function from a (probability) space, i.e. $\Omega^1_i$, to the real numbers $\mathbb{R}$, where the measure is defined on a sigma-field $D_t \subseteq \Omega^1_i$. In other words, for every element (i.e. event of understanding) $\omega_i \in D_t \subseteq \Omega^1_i$, there exists a probability (measure) $\pi^1_i$ associated such that

1. $\pi^1_i(\omega_i) = r_i \in \mathbb{R}$ with probability $\pi_i \quad \forall i$
2. $\pi^1_i(\Omega^1_i) = \bigcup_{i \in \Omega^1_i} r_i \in \mathbb{R}$ with probability $\sum_{i \in \Omega^1_i} \pi_i = 1$

The upper limit of the index set, i.e. $q$, can of course be $\aleph_0$. Different probabilities may be constructed for different levels of generality and abstraction. The relevant issue in this case refers to the nature of the probability space $(\Omega^1_i, D_t, \pi^1_i)$, in particular to the elements of $\Omega^1_i$. In fact, the construction of this space highlights one of the keys in modelling individual novelty. At time $t$, as a new system of dialectical contradictions emerges, the agent learns from it and whatever he/she learns is knowledge that is translated into an (possible) n-fold composite event of knowledge that belongs to $\Omega^1_i$. As it was mentioned previously, these possible events of knowledge are themselves complemented by events of intuition. Hence, the n-fold composite event of knowledge and accompanying intuitions determine the sigma-
algebra $D_t$ that will (under certain restrictions) itself determine the composition of the probability space $(\Omega^i_t, D_t, \pi^i_t)$ at $t+1$. Formally, this space, i.e. $\Omega^i_t$, is defined as

$$\Omega^i_t = \{ u : u \text{ is an understanding} \}$$

$$= \{ u : u \in \Omega^k_t \vee u \in \Omega^m_t \}$$

where

$$\Omega^k_t = \{ u : u \text{ is an event of knowledge} \}$$

$$\Omega^m_t = \{ u : u \text{ is an event of intuition} \}$$

and

$$m(\Omega^m_t) < \infty$$

for any measure $m$ defined on $\Omega^m_t$. Whenever necessary, superscripts as well as subscripts will be used to identify the appropriate agent. All that this definition requires is that an event of understanding at time $t$ must be either knowledge or intuition, and that the subset of intuitions must be finite under any measure. Under this scheme, history up to time $t$ feeds the understanding space (and the underlying sigma-algebra) that determines the value of the random variable $\lambda^i_t$ at $t + 1$. The requirements on the measure of $\Omega^m_t$ will be explained later on in section VI.

The Links

Having defined the sets $H_t$, $F_t$, $K_t$ and $\Omega^i_t$, heterogeneity and diversity of knowledge follows directly through the following operators

$$\Delta: H_t \rightarrow K_t$$

$$\Xi^i: K_t \rightarrow \Omega^i_t$$

$$\Gamma^i: H_t \rightarrow \Omega^i_t$$

termed the dialectical operator ($\Delta$), formal learning ($\Xi^i$) and informal learning ($\Gamma^i$) respectively, that link these sets amongst them. We can describe the composition of an agent’s understanding space in terms of its subspaces. The passage from history to knowledge at $t$, for agent $i$, through formal learning, is thus the composite operator

$$\Theta^i_t = \Xi^i_t \circ \Delta^i_t : H_t \rightarrow \Omega^k_t$$

The passage from history to intuitions is just

$$\Gamma^i_t(H_t) = \Omega^m_t$$

The passage from history to understandings spaces (through any form of learning) is different for every agent. The space of contradictions, i.e. $K_t$, is not, however, different for each agent. This is a result, basically and unequivocally, of the fact that everyone faces the same history. In other words, every one faces the same history and at the same time every one understands it differently (because every one learns differently). If we assume for the time being that $\Omega^i_t$ is indeed a proper probability space and define $TS^i_{t+1}$ to be agent $i$’s time series recorded up to $t + 1$ then we can represent the dynamics of understandings for agent $i$ as
At $t$, agents face a history that allows them to learn (formally and informally). The understandings they gather are transformed into probable influences (of knowledge and intuitions) to be recorded through times series in the next period. The double arrow between $TS_i^{t+1}$ and $F_{t+1}$ is used only to emphasise that, as long as there are more than one agent, $TS_i^{t+1}$ is a proper subset of $F_{t+1}$, i.e. $TS_i^{t+1} \subset F_{t+1}$.

The autopoetic nature of a process driven by the agents’ understandings of the world is fully exposed in diagram 1. At time $t$ more detailed, possibly new, (imperfect) expressions of recorded economic activity are brought forth from the last period. These new expressions enhance and possibly alter the manner in which the agents can learn from the forces that determine economic activity up to $t$. Hence, both history and its recorded facts determine what and how the agents learn. Furthermore, what the agents learn from these new systems may change the nature and composition of their understandings space, however extended or limited is the dialectical knowledge derived thereof. Innovative non-dialectical knowledge may emerge as well. In either case, the agents’ understanding space will ultimately evolve from its previous state at $t - 1$. Hence, the agent’s own record of their economic activity, i.e. the time series, will be influenced through these new expressions of understandings brought forward as events of understandings in $\Omega_i^t$. The outcome of all the agents’ actions, managed through between $t$ and $t + 1$ will be recorded at $t + 1$. These records of history will possibly have evolved from $t$ onto $t + 1$ and hence will have possibly influence the emergence of new dialectical understandings and intuitions about the world. In other words, understandings at $t$ will have generated new understandings at $t + 1$ through knowledgeable economic praxis between $t$ and $t + 1$. The description presented so far attains to individual agents and their respective personalised processes of understanding.

The scheme of understandings generation just describe is general enough to account for all possible sources of heterogeneity. In fact, it brings to the front the need for assumptions about the system’s status quo, its mechanisms of change (including its rate of change) and possible direction of change, i.e. biasness, if there was one. In other words, it brings forth the need for assumption about what Metcalfe (1998) calls
the fundamental determinants of evolutionary change. In terms of individuals this translates into a question of what they can understand, how they can understand and what may ultimately influence their beliefs. It means making assumptions about (individual) distinctions and (collective) similarities in between the agents. This is contrary to the traditional orthodox neoclassical approach to microeconomic modelling where just the opposite is assumed or pursued; that is, individual similarities and collective distinctions.

V Further Characterisations

In this section we discuss some further characterisations concerning the spaces and operators used in the previous section. The proofs of these propositions can be found in the appendix. Whenever necessary we will make certain restrictive assumptions in order to make the discussions manageable and the proofs mathematically consistent.

The definitions provided in the previous section allow us to characterise a fundamental fact, stated before, and claimed by the Theory of Knowledge. That is, that (material) reality, in principle, is a continuum of cognition. Indeed, if we assume that the forces of history, at \( t \), can be counted and that there exists a suitable form of aggregation of these forces (at least theoretically) then we can define a “conjugate” space of history. Let \( \nu \) be the level of aggregation of forces of history (at least theoretically) so that the conjugate space can be defined as

\[
H^*_t,\nu = \{(a, b): (a, b) \text{ are a pair of opposing forces of history considered at the level of aggregation } \nu, \text{ at time } t\}
\]

It will soon become apparent that the issue of the level of aggregation is fundamental in the study of agents’ capabilities in understanding economic activity. For the time being it can be established that the dialectical operator \( \Delta_t \) can thus be defined indistinctively over history, i.e. \( H_t \), or its conjugate, i.e. \( H^*_t,\nu \). If defined over \( H^*_t,\nu \), we can prove the three following complementary propositions:

**Proposition 1** \( K_t \) is closed with respect to unions and interceptions, that is

\[
k_1 \cup k_2 \in K_t \land k_1 \cap k_2 \in K_t \text{ for any } k_1, k_2 \in K_t.
\]

**Proposition 2** The dialectical operator \( \Delta_t: H^*_t,\nu \to K_t \) is a one-to-one map for every fixed level of aggregation \( \nu \).

**Proposition 3** The operator \( \Delta_t: H^*_t,\nu \to K_t \) is continuous over \( H^*_t,\nu \) (not \( t! \)).

What these three propositions, whose proofs can be found in sections A, B and C of the appendix respectively, entail is a characterisation (given our mathematical conceptualisation) on the availability of knowledge to the agents. Indeed, if \( K_t \) is closed under unions and interceptions it implies that \( K_t \) contains all possible “dialectical constructions”. In other words, it contains all possible assemblages that may be structured upon more basic, primitive dialectical contradictions. Additionally \( K_t \) contains, by definition, all the syntheses associated to the (opposing) forces that define reality (dialectically). When combined with the fact that \( K_t \) contains all the constructs thereof (as stated in proposition 1), no matter how infinitesimally small or aggregated are the phenomena in question, \( K_t \) in fact becomes a direct reflection of reality. A sort of abstract mirror upon economic activity where each component in the mirror is a synthesis associated to a pair of (opposing) economic forces. This cognitive reflection, i.e. \( K_t \), indeed makes up a continuum, as claimed by the Theory
of Knowledge. In the process no part of the economic activity has been lost through dialectics, i.e. the dialectical operator is continuous.

In as far as learning is concerned the key issue, in this case, lies in whether there exists or not a lower bound on the level of disaggregation of the forces, i.e. a lower bound on \( \nu \). A continuum of cognition implies necessarily that \( \nu \) must, in principle, be capable of becoming infinitesimally small. It is up to the economic agents to pursue that lower bound. This brings to the front an old idea in a different disguise. That is that agents, in theory, have at their disposal the entirety of (dialectical) reality from which to learn. If they do not hold complete knowledge it is necessarily because they have not, or they cannot, learn it all (except neoclassical agents). The core competence literature recognises this fact and takes it as a corner stone of their approach to the study of the firm, e.g. Prahalad and Hamel (1990), Foss and Knudsen (1996). It has also been recognised by some authors in the economics literature as well. For example, Von Hayek (1937, 1945) spoke of agents incapable of knowing-it-all and hence of agents that hold at most focalised, subjective, bits of knowledge disseminated throughout the economy and coordinated through a pricing mechanism. Penrose (1959) and Richardson (1972) also spoke of limited expressions of knowledge within a firm that only holds what its constituent individuals can learn. The reader by Putterman and Kroszner (1996) contains several investigations on the nature of the firm as well, including Alchian and Demsetz (1972) study on industrial organisation.

Given a fixed level of aggregation \( \nu \) then the syntheses in \( K_\nu \) are necessarily uniquely identified with its constituent forces at that level \( \nu \) of aggregation. If we let \( f_1 = (a_1, b_1), f_2 = (a_2, b_2), f^* = (a^*, b^*), f^\sim = (a^\sim, b^\sim) \) represent four pairs of opposing forces aggregated at level \( \nu \), then we can graph their relationship with their syntheses in the following manner:

![Diagram 2](image)

We can characterise perfectly knowledgeable neoclassical agents based on the above discussions in the following lemma, which we state without proof.

**Lemma.** Perfect knowledge, a la neoclassical, implies that the agents can learn as if \( \nu = 0 \). That is, neoclassical agents hold an infinitely dense, continuum, cluster of knowledge at a single moment in time.
This is a direct consequence of propositions 2, 3 and 4. If an agent learns as if total disaggregation had taken place, i.e. \( \nu = 0 \) at a moment in time, then the agents will have discovered and understood all of the components of economic activity. Hence, the agent will have (formally) learnt it all and thus will hold perfect knowledge of the material economic activity, as defined through dialectics. What in reality takes place, that is, in a non-neoclassical world, is that agents learn from \( K_t \) with \( \nu > 0 \). In other words, they learn with constraints (for the reasons stated at the beginning) thus generating bounded understandings of the world with incomplete beliefs and limited inner logic. That is, they learn in a manner as to induce bounded rationality.

Finally, the present context presents a natural argument against any idealistic conception of understandings in economics. In terms of the description presented here this implies \( \Omega^i_t \) is an open set. Our argument is, again, by contradiction. If \( \Omega^i_t \) were closed then there could exist an event of understanding with a neighbourhood containing some element that might not be an event of understanding. This would imply that there exists a space \( \Omega^* \) such that \( \Omega^i_t \subseteq \Omega^* \). But since \( \Omega^i_t \) incorporates all possible understandings arising from material economic activity (including intuitions) then \( \Omega^i_t \)'s complement (with respect to \( \Omega^* \)) can only be composed of that which has no origin in material reality\(^{11}\). This consideration allows for the possibility of understandings to exist and originate as a possibly idealistic cognisant phenomenon, i.e. economic understandings might originate from non-material foundations, even if dialectical in nature. This possibility, although philosophically valid, attempts against any common sense upon economic activity. Loasby’s (1999, 2000, 2001) insistence on the mental patterns that help individual associate perceived economic phenomena only reaffirms our argument.

Moreover, idealistic knowledge eliminates the possibility of determining some of the core dimensions of the unobserved underlying driving structures of an economic system. The cycle (or loop) defined through the stages of economic activity-abstraction-economic activity is lost. Because of its very nature, idealistic understandings could never provide the necessary context for the emergence of this cycle. Without this cycle one could never guarantee the effects that new emergent understanding will bring onto the industry later on. In other words, history would be lost as a definitive source of understandings. Indeed, some of these effects, and this is the key, are responses to observed past economic activity. Therefore, it is only through this cycle that a causal chain of events can be guaranteed to exist.

The next section is entirely devoted to the construction of \( \pi^i_t \), that is the probability distribution of agent \( i \) at time \( t \). To do this, learning must be examined first so that the required structure, in \( \Omega^i_t \), can be identified. At the same time it will also require making some concrete assumptions about the possible events of knowledge.

VI The Probability Space of Individual Understandings

\(^{11}\) As it was mentioned before intuitions are an acceptable (very much real) form of understanding originated in material reality. In particular, economically relevant intuitions necessarily emanate from material economic activity itself for, otherwise, agents could understand something about economic processes without observing them. That is, out of coincidence which, for present epistemological purposes, is meaningless.
So far the space $\Omega_i^t$ is just the union of the image of $\Xi_i^t$ and $I_i^t$. That is, the space that gathers all events of understanding. These are, forms of understandings concerning the state of affairs up to $t - 1$, achieved through formal and informal learning between periods $t - 1$ and $t$. Even without an explicit recognition of these influences, the definitions above do not, by themselves, guarantee that a probability space could be constructed from $\Omega_i^t$ for the random variable $\lambda_i^t$ at $t$. The question concerning the existence of a Borel $\sigma$-algebra necessary in the construction of a probability is fundamental. For this, further conditions on learning are required. In fact,

**Proposition 4** If the (formal) learning operator, $\Xi_i^t$, is unique (to the agent) at a given moment in time then $\Omega_i^t$ is in fact a $\sigma$-algebra at that moment. Equivalently, for every (individual) learning process there exists a unique $\sigma$-algebra, i.e. $\Omega_i^t$.

**Proof** Consider an arbitrary agent $i$ at an arbitrary moment in time. In order to prove that $\Omega_i^t$ is in fact a $\sigma$-algebra at $t$ it must be proven that

1. $\Omega_i^t \in \Omega_i^t$  
2. For any subset of $\Omega_i^t$, its complement also lies in $\Omega_i^t$. That is, for all $S \in \Omega_i^t$, $S^C \in \Omega_i^t$.  
3. Any (infinitely) countable union of subsets of $\Omega_i^t$ lies in $\Omega_i^t$. That is, if $\{S_i\}$ is a countable collection of subsets of $\Omega_i^t$ then $\bigcup_{i} S_i \in \Omega_i^t$.

Notice that we have used the belong-to symbol, i.e. $\in$, instead of the inclusion symbol, i.e. $\subseteq$, because $\Omega_i^t$ is being considered as a collection of (sub)sets and not a set in itself.

We can trivially ascertain that $\Omega_i^t \in \Omega_i^t$ (as a sigma algebra) since $\Omega_i^t \subseteq \Omega_i^t$. Hence, the rest of the proof deals with 2) and 3).

Consider two arbitrary events in $\Omega_i^t$ such that $\omega_j, \omega_k \neq \emptyset$. If one of these objects were the null set, say $\omega_k$, then we would trivially have $\omega_j \cup \omega_k = \omega_j \notin \Omega_i^t$ for all $j, k$. The same is true if countably infinite unions were considered. Now, consider the pre-image of $\omega_j \cup \omega_k$. There are three possible situations in this case: $\omega_j, \omega_k \in \Omega_i^k$, or $\omega_j, \omega_k \in \Omega_i^m$, or $\omega_j \in \Omega_i^k$ and $\omega_k \in \Omega_i^m$. Let’s see them case-by-case:

If $\omega_j, \omega_k \in \Omega_i^k$ then the pre-image of $\omega_j \cup \omega_k = \Xi_j^i[k_j] \cup \Xi_i^i[k_k]$ is $[\Xi_j^i[k_j] \cup \Xi_i^i[k_k]]^{-1}$ [a] since $\Xi_j^i$ is unique, it defines a one-to-one relationship between syntheses and events of understanding. Henceforth, [a] must equal

$\omega_j \cup \omega_k \in \Omega_i^t$ by virtue of proposition 1. In other words, there exists a class of synthesis of dialectical contradictions, namely $k^* = k_j \cup k_k$ that must have an event of knowledge associated. Whence,

$$\Xi_i^i[k_j \cup k_k] = \Xi_i^i[k_j] \cup \Xi_i^i[k_k] = \omega_j \cup \omega_k = \Xi_i^i[k^*] \in \Omega_i^t$$
This is valid for an infinitely countable union since
\[ \bigcup_{i \in \mathbb{N}} k_i \in K_i \]
If \( \omega_j, \omega_k \in \Omega^{in}_i \) then
\[ \omega_j \cup \omega_k = I_i^j[\text{History}] \cup I_i^k[\text{History}] \]
Now, since intuitions arise out of yet to be discovered protocols of learning we can identify, at least mathematically, each intuition with a different informal operator. That is,
\[ I_i^j[\text{History}] = \omega_i \]
\[ I_i^k[\text{History}] = \omega_k \]
Now, the union of both intuitions must necessarily be an intuition because, otherwise, it would become knowledge. That is, one of the operators \( I_i^j \), \( I_i^k \) would be captured (or transformed into) in \( \Xi_i \). In other words, the union of intuitions could possibly alter the uniqueness of the formal operator at \( t \). Furthermore, the same is true for any infinitely countable union of intuitions. If the (infinitely countable) union were not an intuition it would mean that one of the undiscovered operators would become part of formal learning and hence violate our initial assumption. Hence,
\[ \bigcup_{i \in \mathbb{N}} \omega_i \in \Omega^{in}_i \subseteq \Omega^j_i \]
Finally, if \( \omega_j \in \Omega^k_i \) and \( \omega_k \in \Omega^{in}_i \), that is, one event is knowledge and the other intuition, then
\[ \omega_j \cup \omega_k \in (\Omega^k_i \cup \Omega^{in}_i) = \Omega^j_i \]
for the two events \( \omega_j \) and \( \omega_k \). In general, any infinitely countable unions of events must lie in either \( \Omega^k_i \) or \( \Omega^{in}_i \); that is, it must lie in \( \Omega^j_i \).

Finally, \( \omega \in \Omega^j_i \); then we can define its complement as \( \omega^C = \omega_k \cup \omega_{in} \) where \( \omega_k \) and \( \omega_{in} \) are possibly composite (unions of) events of knowledge and intuitions respectively. By virtue of the previous arguments we have that \( \omega_k \in \Omega^k_i \) and \( \omega_{in} \in \Omega^{in}_i \). Hence,
\[ \omega^C = \omega_k \cup \omega_{in}(\Omega^k_i \cup \Omega^{in}_i) = \Omega^j_i \]
for an arbitrary event of understanding \( \omega \in \Omega^j_i \). Hence, \( \Omega^j_i \) is a Borel \( \sigma \)-algebra. Q.E.D.

What this proposition states is that, as long as there is a uniquely identifiable path between systems of dialectical contradictions and dialectical understandings, then the understanding space itself is a \( \sigma \)-algebra. To use previous terminology it means that \( D_t = \Omega^j_i \). It is a requirement of identification. As long as (formal) learning is unique the events of knowledge that it delivers are identifiable, and hence they can be counted. It does not require the learning operator \( \Xi_i \) to be the same throughout time, it only requires that it not change in between ticks of the (conventional) clock. There could possibly be different learning operator \( \Xi_i \) for every \( t \); however, in the present context the change must take place instantaneously at \( t \).

Since by definition the subspace of intuitions is of finite measure for any measure constructed on \( \Omega^j_i \), then a proper probability measure is guaranteed to exist if the elements in \( \Omega^j_i \) possess certain further characteristics, i.e. essentially topological. A topology in this case constitutes a set of possible understandings. Moreover, different
topologies give rise to different characterisations of the understandings space because, ultimately, they represent different possibilities. This is very much in the spirit of Belianin’s (2000) approach to the question of topologies in economically related abstract sets. In fact, we can prove the following:

**Proposition 5** If formal learning is unique between \( t - 1 \) and \( t \) then the space \((\Omega^i_t, D_t, \pi^i_t)\) is indeed a probability space for any \( t \).

**Proof.** Proposition 4 proved that if learning is unique then \( D_t = \Omega^i_t \) is a indeed a sigma algebra. Whence, the proof will be concerned with the construction of the underlying probability distribution \( \pi^i_t \). The coming arguments are based on Ash(1972), Edwards (1972) and Weir (1974), particularly Edwards (1972).

There are two possible approaches for the construction, the difference being whether the space of understandings are considered to be continuous of discrete. Both require assumptions that somehow restrict the generality of the representation. Nevertheless, they do not compromise in any way whatsoever the previous arguments. The requirements on the appropriate topology will be made precise whenever necessary.

**First Approach.** Assume that \( \Omega^i_t = [a^i_t, b^i_t] = I^i_t \subset \mathbb{R} \) where \( 0 \in [a^i_t, b^i_t] \). Whence, the elements of \( \Omega^i_t \) are the subintervals of \( I^i_t \). These subintervals are closed although they need not be in principle since the difference is constituted by a set of measure zero, i.e. discrete points. Now, consider then following:

1. Let the space of continuous functions over \( I^i_t \) be \( S \) and fix an arbitrary function \( f \in S \). Now, consider a partition of \( I^i_t \):

\[
a^i_t < x_1 < x_2 < \ldots < x_n < b^i_t
\]

where \( p_i = x_i - x_{i-1} \). Call this partition \( P_j = \{p_i\} \).

2. Let

\[
\int_{P_i} f \, dx = \xi_{p_i}
\]

The measure \( dx \) is a Lebesgue measure.

3. Let

\[
\xi^{\text{MAX}}_{P_j} = \max_{P_i} \frac{\xi_{p_i}}{\xi_{p_j}}
\]

This is the maximum value of the integral over the different subintervals defined by the partition \( P_j \). In fact, the quantity defined above is valid for any partition \( P_j \) of \( I^i_t \). Hence,

4. The quantity

\[
\xi^*_f = \max_{P_j} \frac{\xi^{\text{MAX}}_{P_j}}{f}
\]

is well defined. This quantity \( \xi^*_f \) is the maximum value of the area under \( f \) for an interval, given all possible partitions of \( I^i_t \).

Now, define \( F_i : S \to \mathbb{R} \) through which each \( f \in S \) has a real number associated, i.e. \( \xi^*_f \). \( F_i \) is continuous and linear since the process that defines \( F_i \), i.e. steps 1-4, only involve integration and choosing. Additionally, all functions in \( S \) are continuous as well. Finally, choose a topology \( \tau \) so that \( I^i_t \) is both compact and Hausdorff; then the RRT ensures that we can represent \( F_i \) as
where \( \mu \) is a regular Borel measure that depends on the representation. Take careful notice that the integral in the above representation of \( F_i \) is not the integral of step 2. This is so because step 2 is only the integral over a specific subinterval of the partition \( P_j \) whereas \( F_i \) involves maximising the value of the integrals over all possible subintervals, defined by \( P_j \), throughout the family of possible partitions of \( \Omega_1 \), i.e. \( \{P_j\} \). \( F_i \) is, in fact, much more than an integral. It provides an approximation to a central tendency in the functions considered.

In particular, the identity function over this interval \( \text{Id}_i : [a^i_1, b^i_1] \to [a^i_1, b^i_1] \) is continuous under any topology. Then we can ascertain that

\[
F_i(\text{Id}_i) = \int_{[a^i_1, b^i_1]} \text{Id}_i d\mu = m_i
\]

Notice that

\[
\frac{1}{m_i} F_i(\text{Id}_i) = 1
\]

We can now consider the same argument on all the subsets of \( \Omega_1 \) defined as stated at the beginning. Then the identity function over a subinterval \( \text{sub}^i_1 \), denoted \( \text{Id}_i(\omega_j) : \text{sub}^i_1 \to \text{sub}^i_1 \) is again bounded and continuous and hence the RRT is applicable. In fact we can define a probability measure for (any) \( \omega_j = [\alpha_j \beta_j] \) (where of course \( a^i_1 \leq \alpha_j \leq \beta_j \leq b^i_1 \)) by considering

\[
\pi^i_j(\omega_j) = \frac{1}{m_i} F_i(\text{Id}_i(\omega_j)) = \frac{1}{m_i} \int_{[a^i_1, b^i_1]} \text{Id}_i(\omega_j) d\mu = \alpha_j < 1 \tag{b}
\]

If we define non-intersecting events of understanding to be \( \omega_j \cap \omega_l = [\alpha_j \beta_j] \cap [\alpha_l \beta_l] = \phi \) then

\[
\pi^i_j(\omega_j + \omega_l) = \frac{1}{m_i} F_i(\text{Id}_i(\omega_j + \omega_l)) = \frac{1}{m_i} \int_{[a^i_1, b^i_1]} \text{Id}_i(\omega_j) d\mu + \frac{1}{m_i} \int_{[a^i_1, b^i_1]} \text{Id}_i(\omega_l) d\mu = \alpha_j + \alpha_l
\]

In fact if \( \bigcap_{i \in \tilde{J}} \text{sub}^i_1 = \phi \) where \( \tilde{J} \) is some indexing set for \( [a^i_1, b^i_1] \) then

\[
\pi^i_j(\bigcup_{i \in \tilde{J}} \omega_i) = \pi^i_j(\sum_{i \in \tilde{J}} \text{sub}^i_1) = \frac{1}{m_i} \sum_{i \in \tilde{J}} \int_{[a^i_1, b^i_1]} \text{Id}_i(\omega_i) d\mu = 1 \tag{c}
\]

where \( \tilde{J} \) is the \( i \)th member of the indexing set \( \tilde{J} \). Finally, if \( \{\text{sub}^i_1\} \) are such that they do not reduce integration to a set of measure zero, i.e.

\[
\pi^i_j(\sum_{i \in \tilde{J}} \text{sub}^i_1) = 0
\]

then, \( \pi^i_j \) has finite variation given by

\[
V(\pi^i_j) = \int_{[a^i_1, b^i_1]} \text{Id}_i d\mu = \|F_i\|
\]

Equations [E.1] and [E.2] guarantee that \( \pi^i_j \) is indeed a well defined probability measure over \( \Omega_1 = [a^i_1, b^i_1] \). This construction specifies the values in probability for each event of understanding through [b]. Conceptually, it represents an approximation of the central tendency in the occurrence of the event.
The probability distribution $\pi^t_{\text{system}}$ for the product space, here termed $\Omega^t_{\text{system}} = \Omega^t_1 \times \Omega^t_2 \times \cdots \times \Omega^t_n$, is directly induced by each component. Indeed, if each agent’s understanding space $\Omega^t_i = [a^i_1 \ b^i_1]$ has an associated topology $\tau_i$ that makes it compact and Hausdorff then the product topology will make $\Omega^t_{\text{system}}$ compact and Hausdorff (by virtue of Tychonoff’s theorem and the natural separation of points in $\Omega^t_{\text{system}}$). The analogous linear functional $F_{\text{system}}$ is now defined the product space $S_{\text{system}} = S_1 \times S_2 \times \cdots \times S_n$ so that

$$F_{\text{system}} : S_{\text{system}} \to \mathbb{R}^n$$

where

$$F_{\text{system}} = F_1 \times F_2 \times \cdots \times F_n$$

Each one of the components of $F_p$ defines the marginal probability upon that particular agent. The probability distribution $\pi^t_{\text{system}}$ is conditioned on the individual probability distributions and is thus defined as

$$\pi^t_{\text{system}}(\theta_j | \Omega^t_1, \Omega^t_2, \ldots, \Omega^t_n) = \sqrt{\sum_i (\pi^t_i(\omega_{j,i}))^2} = \sqrt{\sum_i \left( \frac{1}{m_i} F_i(\text{Id}_i(\omega_{j,i})) \right)^2} = \sqrt{\sum_i \left( \frac{1}{m_i} \int \text{Id}_i(\omega_{j,i}) d\mu \right)^2}$$

where the event of understanding in the entire system is the composite event

$$\theta_j = \omega_{j,1} \times \omega_{j,2} \times \cdots \times \omega_{j,n}$$

The value of each one of the components of $\pi_{\text{system}}^t(\theta_j | \Omega^t_1, \Omega^t_2, \ldots, \Omega^t_n)$ lies, by construction, between 0 and 1. Geometrically, the probability density may be graphed as an n-dimensional cube.

Second Approach. Assume that $\Omega^t_i$ is countable. Hence, there exists an isomorphism between $\Omega^t_i$ and the natural numbers $\mathbb{N}$ that identifies each event of understanding with an (ordered) natural number. Whence, we can write $\Omega^t_i = \{\omega_j\}$. This identification also leads naturally to identification with the rational numbers $\mathbb{Q}$ as well (because of the isomorphism between $\mathbb{N}$ and $\mathbb{Q}$). In any case, consider an arbitrary function $\varphi : \Omega^t_i \to \mathbb{R}$ such that

$$\varphi(\omega_j) = r_j$$

where $r_j$ are discrete real numbers. Now, we can order the $r_j$’s using the induced order from the real numbers, i.e. $\leq$ so that $T_i = \{r_j\}$ is an ordered countable set. Because of this order $T_i$ thus has a minimum and a maximum value. Call them $r^{\text{MIN}}$ and $r^{\text{MAX}}$ respectively. Also, $T_i$ is trivially $\sigma$-algebra because of the identificability with $\Omega^t_i$. Only unions and complements of elements in $T_i$ that can be “traced back” to $\Omega^t_i$ have any meaning, i.e. are defined. Hence, $T_i$ is necessarily a $\sigma$-algebra.

Consider now,

1. Let $f \in S^t_i = \{\text{set of continuous functions on } T_i\}$ be an arbitrary function. The existence of this set is equivalent to saying that there exists a topology $\delta$ that gives rise to a set (actually a Hilbert space of functions) of continuous functions. Hence, define
\[
m(f(r_k)) = \left| f(r_k) - r^{\text{MIN}} \right|
\]
so that
\[
m(f(r_k) + f(r_i)) = \left| \max \{ f(r_k), f(r_i) \} - r^{\text{MIN}} \right|
\]

2. Let
\[
m^*_f = \max_{r_k} \{ m(f(r_k)) \}
\]
Hence, for every \( f \in S^* \) there exists an associated real number, i.e. \( m^*_f \). We can then define the following functional
\[
F_i(f) = m^*_f
\]
\( F_i \) is not just the difference between numbers but actually involves choosing the biggest difference amongst them as determined by \( T_i \) and \( f \). It is a continuous linear functional since given a fixed set \( T_i \) a sequence \( \{ g_i \} \) can always be defined to approximate any \( f \in S^* \). This is so because of the Cauchy property of the real numbers and any subset thereof. The Cauchy property states that given any fixed real number there exists a convergent sequence of real numbers whose difference amongst the members of the sequence shrinks the further the convergence. Finally, any \( f \in S^* \) is defined over the real numbers and has a range of real numbers. Hence, choose a topology \( \tau \) over \( T_i \) that: 1) maintains the continuity of any \( f \in S^* \), 2) makes \( T_i \) compact and Hausdorff. Then, the RRT, allows us to represent \( F_i \) as
\[
F_i(f) = \int f \, d\mu
\]
In particular,
\[
F_i(\text{Id}_{T_i}) = \int \text{Id}_{T_i} \, d\mu = \left| r^{\text{MAX}} - r^{\text{MIN}} \right| = \xi_i
\]
so that
\[
\frac{1}{\xi_i} F_i(\text{Id}_{T_i}) = 1
\]
Therefore, if \( r_i < r^{\text{MAX}} \) then
\[
F_i(\text{Id}_{T_i}(r_i)) = \int \text{Id}_{T_i}(r_i) \, d\mu = \left| r_i - r^{\text{MIN}} \right|
\]
We can define a probability measure for the event \( \omega_i \) as
\[
\pi^i_1(\omega_i) = \frac{1}{\xi_i} F_i(\text{Id}_{T_i}(r_i)) = \frac{1}{\xi_i} \int \text{Id}_{T_i}(r_i) \, d\mu = \frac{\left| r_i - r^{\text{MIN}} \right|}{\xi_i} = \alpha_i < 1
\]
If \( \omega_i \) and \( \omega_k \) are two non-intersecting events then \( r_i \cap r_k = \phi \) then we can define the probability of the union as
\[
\pi^i_1(\omega_i \cup \omega_k) = \frac{1}{\xi_i} F_i(\text{Id}_{T_i}(r_i \cup r_k)) = \frac{1}{\xi_i} \int \text{Id}_{T_i}(r_i \cup r_k) \, d\mu = \max \left\{ \frac{1}{\xi_i} \int \text{Id}_{T_i}(r_i) \, d\mu, \frac{1}{\xi_i} \int \text{Id}_{T_i}(r_k) \, d\mu \right\}
\]
From this we can ascertain, using the identifiability of \( \Omega_i \) with \( T_i \), that if \( \bigcap \omega_j = \phi \) then
\[
\pi^i_1(\bigcup \omega_i) = \frac{1}{\xi_i} F_i(\text{Id}_{T_i}(\bigcup T_i)) = \frac{1}{\xi_i} \int \text{Id}_{T_i}(\bigcup T_i) \, d\mu = \sum \alpha_i \leq 1
\]
In particular, if \( \bigcup T_i = T_i \) then
\[ \pi^i_t(\bigcup \omega_j) = 1 \]  

Equations [c], [d], [e] and [f] ensure that \( \pi^i_t \) is a proper probability distribution for \( \Omega^i_t \) at time \( t \).

Just as in the previous construction, the topological properties of the product space \( \Omega^i_{\text{system}} = T_1 \times T_2 \times \cdots \times T_n \) follow directly from the properties of each one its component. A (product) topology \( \tau_{\text{system}} \) that makes \( \Omega^i_{\text{system}} \) Hausdorff and compact is one determined by the product of those topologies \( \tau_i \) that make each \( \Omega^i_t \) Hausdorff and compact. Again, the analogous linear functional is defined as

\[
F_{\text{system}} : S_p^* \to \mathbb{R}^n
\]

where

\[
S_p^* = S^*_1 \times S^*_2 \times \cdots \times S^*_n
\]

and

\[
F_{\text{system}} = F_1 \times F_2 \times \cdots \times F_n
\]

The probability distribution for an arbitrary composite event \( \theta_j \) in \( \Omega^i_{\text{system}} \) is thus

\[
\pi^i_{\text{system}}(\theta_j | \Omega^i_1, \Omega^i_2, \ldots, \Omega^i_n) = \sqrt{\sum_i \left(\pi^i_i(\omega_{j,i})\right)^2}
\]

\[
= \sqrt{\sum_i \left(\sum_{\xi_j} \frac{1}{\xi_j} F_i(I_{\xi_j}(r_{j,i}))\right)^2}
\]

\[
= \sqrt{\sum_i \left(\sum_{\xi_j} \frac{1}{\xi_j} \int I_{\xi_j}(r_{j,i}) d\mu\right)^2}
\]

\[
= \sqrt{\sum_i \left(\sum_{\xi_j} \frac{\left|f_{j,i} - r_{j,i}^{\text{MIN}}\right|}{\xi_j}\right)^2}
\]

\[
= \sqrt{\sum_{\xi_j} (a_{j,i})^2}
\]

where \( \theta_j = \omega_{j,1} \times \omega_{j,2} \times \cdots \times \omega_{j,n} \) and \( a_{j,k} \leq 1 \) for \( k = 1, 2, \ldots, n \). Again, each component \( \frac{f_{j,k} - r_{k}^{\text{MIN}}}{\xi_k} \) defines the marginal probabilities of \( \pi^i_{\text{system}} \) for each \( k = 1, 2, \ldots, n \).

Q.E.D.

It can be seen that a number of different probabilities can be constructed. We have only shown two in order to emphasise the role of learning and understandings in general in the process of formation of the understanding space. Moreover, assumptions about the agents’ understanding spaces and their relation to the real numbers are not so restrictive either; especially in light of the equivalence of all measures in \( \mathbb{R} \). The crystallizations of understandings are always, at any time, real numbers simply because they are part of measured economic activity. Hence, it is by
no means arbitrary to conceptualise the agents’ understandings spaces, in as far as modelling is concerned, as subsets of $\mathbb{R}$. As stated previously, this particular construction of the probability measure was chosen for $\Omega_i^t$, which is ultimately equivalent to a Lebesgue measure (because of the need to crystallise understandings as measured recordings), essentially to highlight the dynamics of understandings. Having stated this, the constructions are by no means unique. Notwithstanding, all constructions are equivalent in as much as knowledge and intuitions are concerned.

The probability distribution, constructed at time $t$, based on the dynamics of understandings generation is of course $\pi_i^t$, the probability just constructed, i.e.

$$\pi_i^t : \Omega_i^t \to [0,1]$$

This probability, i.e. [2], is the underlying probability measure of the random variable $\lambda_i^t$ in our general statistical specification of section I. Moreover, they may have different specific forms as far as empirical issues are concerned. However, for the present context, it is not a concern simply because the basis of this probability are not (possibly) repeated experiments nor combinatorial deducts. It is not a subjective measure of anything either. They represent the effects of history, as channelled through learning, borne by the different agents. Anything else concerning the nature of these probabilities is an assumption. In particular, for agent $i$, we have that

$$\lambda_i^t(\omega) = r_{\omega} \in \mathbb{R}$$

where

$$\pi_i^t(\omega) = x \in [0,1], \forall \omega \in \Omega_i^t$$

given, of course, that a suitable topology $\tau$ has been chosen to represent the set of possible events of understanding by the agent at $t$. In this case, then, the event $\omega$ takes place as $r_{\omega} \in \mathbb{R}$ with probability $0 \leq x \leq 1$. In terms of our econometric specification we have

$$h_i^t = f(h_i^t, z_i^t) + r_{\omega^*}$$

with probability $x_{\omega^*}$ for some event of understanding $\omega^*$ acquired at $t - 1$; that is, $\lambda_i^t(\omega^*) = r_{\omega^*}$ and $\pi_i^t(\omega^*) = x_{\omega^*}$.

The construction of the probability distributions associated to the random disturbance term $\lambda_i^t$ emphasised a couple of points previously hinted at. First of all, it made explicit the requirement that learning be unique in between measurements of economic activity, i.e. between ticks of the conventional clock. Second, the requirement, at every $t$, on the topology of $\Omega_i^t$ highlighted the fact, recognised in the core competence literature, that not all forms of understandings do indeed become relevant in individual decision making procedures. There are capabilities constraints inherent in the internal workings of individuals and firms that restrict the availability in use of understandings to agents and firms. The process was also seen to be entirely subjective in respect to learning. Individual (or firm) limitations on understanding processing capabilities were reflected in the spread of possible topologies to consider. Hence, we can always interpret limitations in individual understandings as arising from either learning or limitations in handling understandings (thus defining coarser and coarser topologies). There will always exist at least one topology, i.e. the coarsest, that will guarantee the applicability of the RRT at each $t$. However, this case is very uninteresting and trivial. Agents normally operate anywhere in a position between the most trivial topology, i.e. the coarsest, and the finest, i.e. the one that
contains all possible subsets of $\Omega_i^{12}$. In general, the topologies associated to each $\Omega_i$ specify the possibilities of understandings. The RRT specifies their probable occurrence.

VII Conclusions

It was argued that the passage from history to understandings spaces was seen to be essentially a two-fold sequential process. First, an elaboration of a system of materially-based dialectical contradictions takes place. Second, formal individual learning takes place over this system. The result is knowledge, which is complemented through informal learning that determines intuitions. Moreover, it is at this latter stage, the learning stage, in which heterogeneity is guaranteed. Knowledge becomes an individual human attribute once it is internalised. All forms of individual understandings are gathered in the agents’ understandings space.

History, it was further argued, exposes new facts, which are incorporated to the “stock” of information. Ergo, time presents itself as the required vertex through which novel understandings emerge. These new possible understandings, depending on each individual agent, may or may not become new probable events of understandings. It was proven that if

- Learning is unique between two recorded observations
- A suitably topology in the space of understandings is chosen

then a proper probability space for the random variable $\lambda_i$ in [1] can be constructed. This implied that not all possible forms of understandings could become events of understandings. Concretely, for every suitable topology there is a set of (possibly distinct) events of understandings and therefore different probability distributions. A suitable topology was seen to be one that made the agent’s understandings space a Borel $\sigma$-algebra.

In summary, the paper proved that for every different learning procedure and every different topology (and any combination thereof) there exists a distinct probability distribution associated to $\lambda_i$. Moreover, in order to construct this probability distribution learning must be unique between any two consecutive records of economic activity. That is, for learning to evolve and probability spaces to be constructible, learning must not change between two consecutive recordings and a suitable topology has to be chosen as well. In this way knowledge and intuitions form the events of the underlying probability space of the random variable in [1].

12 This topology is called the Power Set of $\Omega_i$ and it is denoted as $P(\Omega_i)$. 
References


APPENDIX

A. Proposition 1

Consider two arbitrary contradictions from $K_t$, say $k_i$ and $k_j$. Beyond the expression’s digital symbolism, the union (of anything) captures the idea of two aggregated entities, in this case of two contradictions. When put together these two contradictions can represent either a more extensive system than the one defined by each of the contradictions separately\(^{13}\) or a new more aggregated (and less specific) contradiction. Furthermore, two facts stand out: there exist at least four (opposing) forces that define the new contradiction (or system) and there exists a path that concatenates $k_i$ and $k_j$ together (because of the connectivity of $K_t$). Let $(a, b) \leftrightarrow k_i$ and $(d, e) \leftrightarrow k_j$ be the four forces that underlie the new (aggregated) context. If the aggregation of forces is (arbitrarily) defined to be $(a, b) + (d, e) = (a \oplus d, b \oplus e)$ where $\oplus$ is some form of aggregation of forces (at least conceptually) then what the union of contradictions, i.e. $k^* = k_i \cup k_j$, represents is simply a new unity of opposites where the thesis and antithesis are $a \oplus d$ and $b \oplus e$ respectively. The particular definition of the aggregation of forces just presented only attempts to maintain consistency in the notation and conceptual coherence. However, it need not be the only form of aggregation. Moreover, the case for the interception of contradictions, i.e. $k_i \cap k_j$, is even more direct. Indeed, if the interception in non-empty, i.e. $\emptyset \neq k_i \cap k_j$, then the interception of either the theses, i.e. $a$ and $c$, or the antitheses, i.e. $b$ and $d$, or both is non-empty. That is, either $a \cap c \neq \emptyset$ or $b \cap d \neq \emptyset$ or both. Hence, the interception $k^\cap = k_i \cap k_j$ of both contradictions is itself a contradiction defined by the (opposing) forces $(a \cap c \setminus a \oplus c, b \cap d \setminus b \oplus d)$. In both case the forces are defined to be what is mutual minus what is not shared. In this case, the commonality in the theses and antitheses is what defines the possibly “smaller” contradiction $k^\cap$. Finally, notice a subtlety in the manner that the elements of $K_t$ are handled: aggregation is always indirect and not necessarily unique whereas interception is not. The reason is that the interception of forces is always readily defined in terms of commonality whereas aggregation of forces requires a mechanism, or rule, of aggregation.

B. Proposition 2

Our argument is intrinsically by contradiction. If $\Delta_t$ is defined over $H^\ast_{t,\nu}$, i.e. $\Delta_t: H^\ast_{t,\nu} \rightarrow K_t$, and it is not a one-to-one map then there exists at least one $k \in K_t$ such that $[k]^{-1} = [\Delta_t(\varsigma)]^{-1}$ is not a singleton. That is, the preimage of $k$ is not a singleton. To put it differently, there exist at least 2 distinct pairs of economic opposing forces, at $t$, i.e. $\varsigma \neq \varsigma^\ast$, such that $\Delta(\varsigma) = k = \Delta(\varsigma^\ast)$. But, since the level of aggregation $\nu$ of forces is fixed and both $\varsigma$ and $\varsigma^\ast$ give rise to the same synthesis, i.e. $k$, then necessarily they must both represent the same aggregate of forces. That is, $\varsigma = \varsigma^\ast$. This is a contradiction since $\Delta_t$ was assumed not to be one to one. Whence, it must be one to one for a fixed level of aggregation $\nu$.

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\(^{13}\) In strict rigour, what the two aggregated contradictions, in fact, form is a subsystem of the entire system.
C. Proposition 3

In order to prove the continuity of $\Delta_t$ it is sufficient to consider an arbitrary convergent sequence $\{\zeta_t\}$ of events in $\Delta_t$'s domain. If $\{\Delta_t(\zeta_t)\}$ is convergent as well then $\Delta_t$ is continuous. As before $\Delta_t$ is defined over $H^*_t,\nu$ (this means that $\zeta_t$ is a pair of opposing economic forces at $t$). The level $\nu$ of aggregation of economic forces induces a natural topology on $H^*_t,\nu$ that may be termed $\tau_{t,\nu}$. This topology is, by construction and nature, countable and is composed of all of the countable subsets of $H^*_t,\nu$ at a fixed level $\nu$ of aggregation. This topology is the finest topology induced by the level of aggregation $\nu$. Also, it should be clear that this (finest) topology changes with $\nu$. Hence, if $\{\zeta_t\}$ is a convergent sequence then it means that the sequence has a cluster point under some topology, possibly coarser than $\tau_{t,\nu}$. Call this topology $\rho_{t,\nu}$, i.e. $\rho_{t,\nu} \subseteq \tau_{t,\nu}$, and the sequence’s cluster point $\zeta_t$. That is, for all neighbourhoods $N(\zeta_t)$ of $\zeta_t$, under $\rho_{t,\nu}$, there exists at least one $\zeta_t' \in \{\zeta_t\}$ different from $\zeta_t$ such that $\zeta_t' \in N(\zeta_t)$.

Now, since $\Delta_t$ is one to one (for the fixed level of aggregation $\nu$) then $\Delta_t(\zeta_t), \{\Delta_t(\zeta_t)\}$ and $\{\Delta_t(p)\}$, where $p \in \rho_{t,\nu}$ are the components of the topology, are all well defined quantities in $K_t$. In fact, the collection $\{\Delta_t(p)\}$ forms a topology for $K_t$ (because $\Delta_t$ is one to one). This topology is, again, possibly coarser than the topology defined by the image, under $\Delta_t$, of $\tau_{t,\nu}$. Call the topology in $K_t$ that arises from the image of $\rho_{t,\nu}$ under $\Delta_t$, $\delta(\rho_{t,\nu})$. This topology $\delta(\rho_{t,\nu})$ is also countable and completely identified in $H^*_t,\nu$. For this topology every neighbourhood $N(\Delta_t(\zeta_t))$ of $\Delta_t(\zeta_t)$, under $\delta(\rho_{t,\nu})$, contains at least one member $\Delta_t(\zeta_t') \in \{\Delta_t(\zeta_t)\}$, different from $\zeta_t$. To see this, it is enough to consider the preimages of the objects in $K_t$. That is, pick an arbitrary neighbourhood of $N(\Delta_t(\zeta_t))$ and consider its preimage $[N(\Delta_t(\zeta_t))]^{-1}$. Since $\Delta_t$ is one to one and $\delta(\rho_{t,\nu})$ is discrete topology then $[N(\Delta_t(\zeta_t))]^{-1}$ must be a neighbourhood of $\zeta_t$. Hence, it must contain a member of the series $\{\zeta_t\}$, i.e. there exists at least one $\zeta_t' \in \{\zeta_t\}$ such that $\zeta_t' \in [N(\Delta_t(\zeta_t))]^{-1}$. This implies that there exist an open set in $[N(\Delta_t(\zeta_t))]^{-1}$, say $O_{pre}$, such that $\zeta_t' \in O_{pre}$. Since $\Delta_t$ is one to one and $[N(\Delta_t(\zeta_t))]^{-1}$ is a discrete set we therefore have $\Delta_t(\zeta_t') \in \Delta_t(O_{pre}) \subseteq N[\Delta_t(\zeta_t)]$.

That is, given an arbitrary neighbourhood $N(\Delta_t(\zeta_t))$ of $\zeta_t$, under $\delta(\rho_{t,\nu})$, then there exists at least one $\Delta_t(\zeta_t') \in \{\Delta_t(\zeta_t)\}$, different from $\zeta_t$, such that $\Delta_t(\zeta_t') \in N(\Delta_t(\zeta_t))$. Hence, $\Delta_t(\zeta_t)$ is a cluster point of $\{\Delta_t(\zeta_t)\}$, under the “induced” topology $\delta(\rho_{t,\nu})$. That is, $\{\Delta_t(\zeta_t)\}$ converges and $\Delta_t$ is thus continuous for the level for aggregation $\nu$. 